Closing Tues: 12.5(2)(3), 12.6
Closing Thurs: 13.1, 13.2

## *Entry Task*

1. Find the equation for the plane through $\overset{P}{(2,0,0)}$, $\overset{Q}{(0,3,0)}$, $\overset{R}{(0,0,6)}$.
2. Find the equation of the line through $\overset{A}{(0,0,1)}$ and $\overset{B}{(5,4,3)}$
3. Find the intersection of this plane and this line.

$\boxed{1}$ POINT : $(2,0,0)$

$\overrightarrow{PQ} = \langle -2, 3, 0\rangle$ = PARALLEL TO PLANE

$\overrightarrow{PR} = \langle -2, 0, 6\rangle$ = PARALLEL TO PLANE

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{vmatrix}$$

$$= (18 - 0)\vec{i} - (-12 - 0)\vec{j} + (0 - -6)\vec{k}$$

$$= \langle 18, 12, 6\rangle$$

$$18(x-2) + 12(y-0) + 6(z-0) = 0$$

$\div 6$

$$3(x-2) + 2y + z = 0$$

$$3x - 6 + 2y + z = 0$$

$$\boxed{3x + 2y + z = 6}$$

NOTE: Check that all three points work!!!

$3\cdot 2 + 2\cdot 0 + 0 = 6$ ✓
$3\cdot 0 + 2\cdot 3 + 0 = 6$ ✓
$3\cdot 0 + 2\cdot 0 + 6 = 6$ ✓

$\boxed{2}$ POINT: $(0,0,1)$ DIRECTION: $\overrightarrow{AB} = \langle 5, 4, 2\rangle$

$$\boxed{\begin{aligned} x &= 0 + 5t \\ y &= 0 + 4t \\ z &= 1 + 2t \end{aligned}}$$

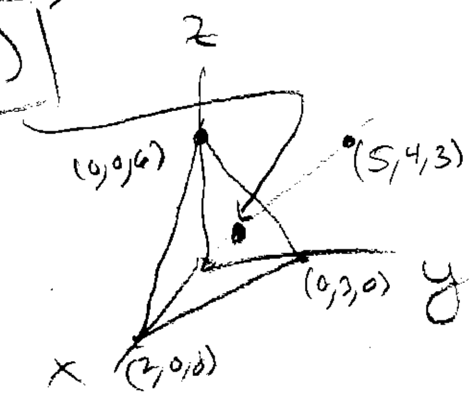
$\boxed{3}$ $3(5t) + 2(4t) + (1+2t) \overset{?}{=} 6$

$\Rightarrow 15t + 8t + 1 + 2t = 6$

$\Rightarrow 25t = 5 \Rightarrow t = 1/5$

THUS, $x = 1$, $y = 4/5$, $z = 1 + 2/5 = 7/5$

$$\boxed{(x,y,z) = (1, 4/5, 7/5)}$$



## *LINES*

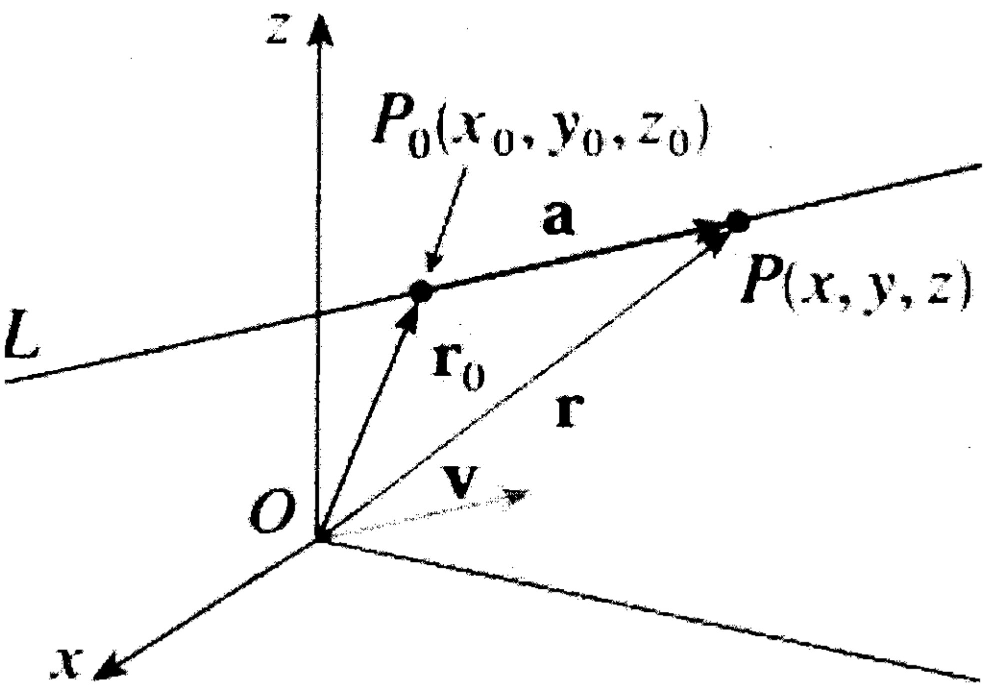
Find **<u>a direction vector</u>** and **<u>a point</u>**

**1.** $\boldsymbol{v} = \langle a, b, c \rangle$ *direction vector*

**2.** $\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ *position vector*

All points (x,y,z) on the line satisfy:

$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \mathrm{t}\, \langle a, b, c \rangle$




## *PLANES*

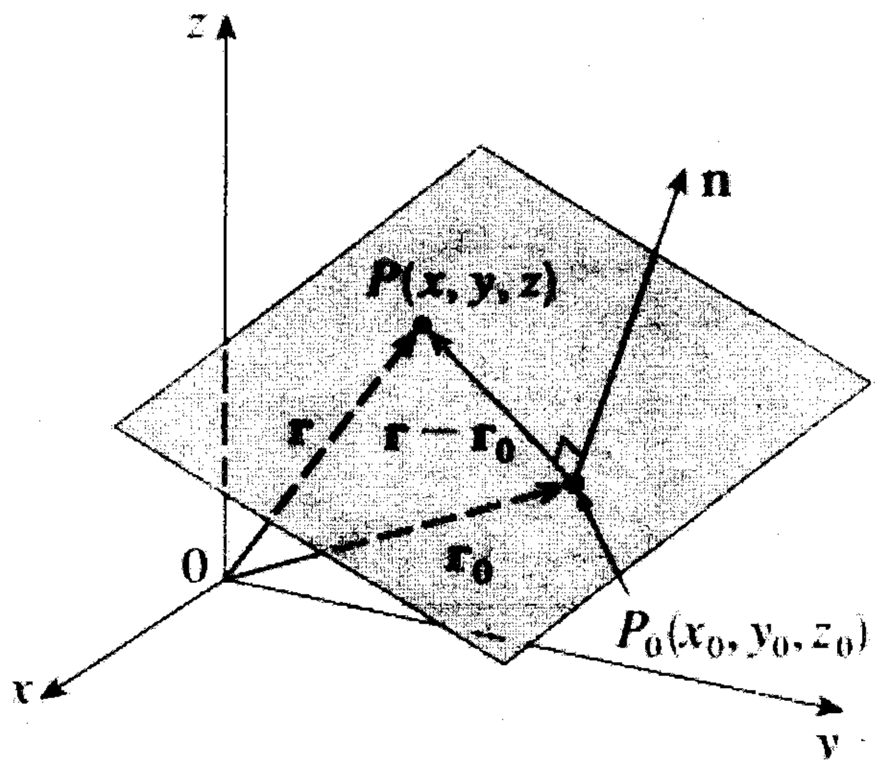
Find **<u>a normal vector</u>** and **<u>a point</u>**

**1.** $\boldsymbol{n} = \langle a, b, c \rangle$ *normal vector*

**2.** $\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ *position vector*

All points (x,y,z) on the plane satisfy:

$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$



**12.5 Summary**

"Find the equation of a line…"

*Step 1*: Write
$x = x_0 + at, y = y_0 + bt, z = z_0 + ct.$

*Step 2*: Write down all the given information. Find a Point and a Direction.

| To find equations for a line | | |
|---|---|---|
| Info given? | Find two points | |
| Done. | $\vec{v} = \overrightarrow{AB}$ (subtract components) | $\overrightarrow{r_0} = \vec{A}$ |

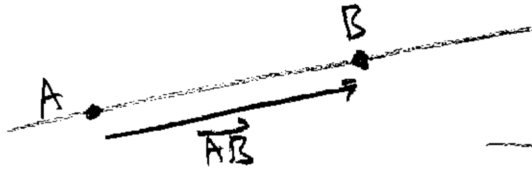
"Find the equation of a plane…"

*Step 1*: Write
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

*Step 2*: Write down all the given information. Find a Point and a Normal.

| To find the equation for a plane | | |
|---|---|---|
| Info given? | Find three points | |
| Done. | Two vectors parallel to the plane: $\overrightarrow{AB}$ and $\overrightarrow{AC}$ | |
| | $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ | $\overrightarrow{r_0} = \vec{A}$ |

1. Find an equation for the line that goes through the two points $A(1,0,-2)$ and $B(4,-2,3)$.




$\boxed{1}$ POINT: $(1,0,-2)$ (OR $(4,-2,3)$)

$\boxed{2}$ DIRECTION: $\vec{AB} = \langle 3,-2,5 \rangle$ (OR $\vec{BA}$)

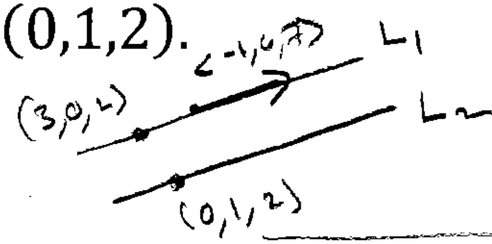
$$\boxed{\begin{aligned} x &= 1+3t \\ y &= 0-2t \\ z &= -2+5t \end{aligned}}$$

CHECK: ARE THE PTS ON THE LINE?

$(1,0,-2) \leftrightarrow t=0$ ✓

$(4,-2,3) \leftrightarrow t=1$ ✓

2. Find an equation for the line that is parallel to the line $x = 3 - t$, $y = 6t, z = 7t + 2$ and goes through the point $P(0,1,2)$.

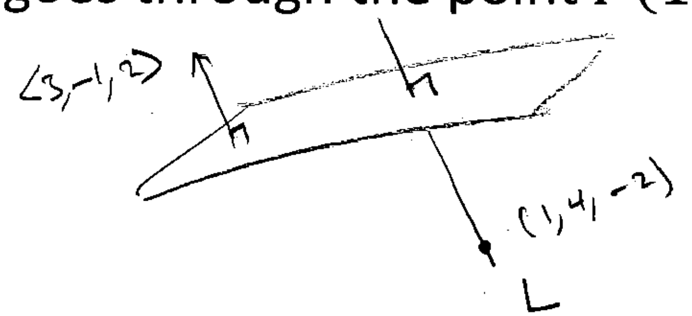



$\boxed{1}$ POINT: $(0,1,2)$

$\boxed{2}$ PARALLEL LINES HAVE PARALLEL DIRECTION VECTORS! SO YOU CAN USE $\langle -1,6,7 \rangle$

$$\boxed{\begin{aligned} x &= 0-t \\ y &= 1+6t \\ z &= 2+7t \end{aligned}}$$

3. Find an equation for the line that is orthogonal to $3x - y + 2z = 10$ and goes through the point $P(1,4,-2)$.




$\boxed{1}$ POINT: $(1,4,-2)$

$\boxed{2}$ THE VECTOR $\langle 3,-1,2 \rangle$ IS ORTHOGONAL TO THE PLANE AND THE LINE IS ORTHOGONAL TO THE PLANE. SO SAME DIRECTION!

$$\boxed{\begin{aligned} x &= 1+3t \\ y &= 4-t \\ z &= -2+2t \end{aligned}}$$

4. Find an equation for the line of intersection of the planes

$$5x + y + z = 4 \text{ and}$$

$$10x + y - z = 6.$$

FIND ANY TWO POINTS ON THE INTERSECTION!

step 1) Combine

(i) $5x + y + z = 4 \Leftrightarrow z = \underbrace{4 - 5x - y}$

(ii) $10x + y - z = 6$

OR JUST ADD CORRESPONDING SIDES (SAME THING)

$$15x + 2y = 10$$

step 2) Pick ANY VALUE FOR x OR y AND SOLVE FOR COORDINATE

$x = 0 \Rightarrow 2y = 10 \rightarrow y = 5 \Rightarrow z = 4 - 5(0) - 5 = -1$

$A(0, 5, -1)$ CHECK! ✓

$y = 0 \Rightarrow 15x = 10 \rightarrow x = \frac{10}{15} = 2/3 \Rightarrow z = 4 - \frac{10}{3} - 0$

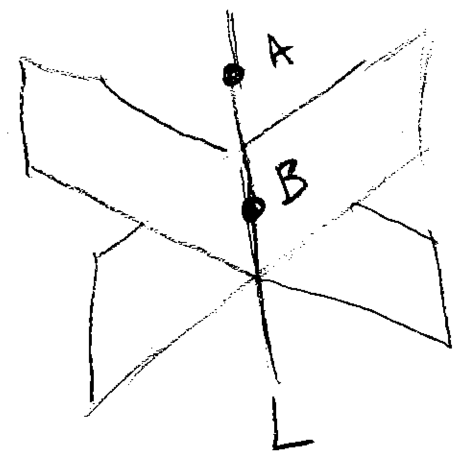
$z = 2/3$

$B(\frac{2}{3}, 0, 2/3)$ CHECK! ✓

POINT: $(0, 5, -1)$

DIRECTION: $\overrightarrow{AB} = \langle \frac{2}{3}, -5, 5/3 \rangle$

$$\boxed{\begin{aligned} x &= 0 + \tfrac{2}{3}t \\ y &= 5 - 5t \\ z &= -1 + \tfrac{5}{3}t \end{aligned}}$$



1. Find the equation of the plane that goes through the three points A(0,3,4), B(1,2,0), and C(-1,6,4).

1 POINT: $(0,3,4)$ (OR B OR C)

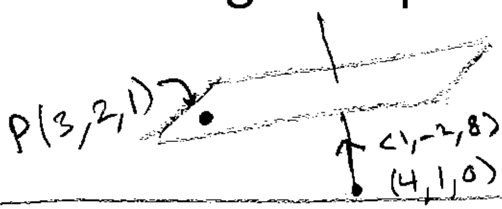
2 NORMAL: $\vec{AB} = \langle 1,-1,-4\rangle$, $\vec{AC} = \langle -1,3,0\rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -4 \\ -1 & 3 & 0 \end{vmatrix} = (0 - -12)\vec{i} - (0-4)\vec{j} + (3-1)\vec{k}$$
$$= \langle 12, 4, 2\rangle$$

CHECK ✓
$12-4-8=0$ ✓
$-12+12+0=0$ ✓

$$\boxed{12x + 4(y-3) + 2(z-4) = 0}$$ CHECK PTS!

2. Find the equation of the plane that is orthogonal to the line $x = 4 + t, y = 1 - 2t, z = 8t$ and goes through the point P(3,2,1).



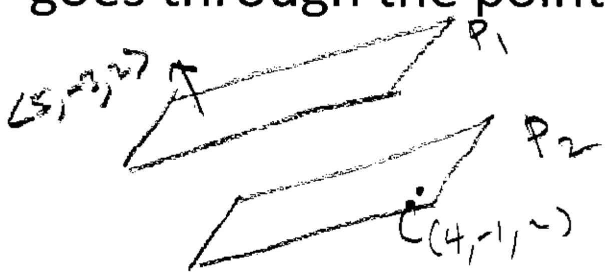

1 POINT: $(3,2,1)$

2 LINE GOES IN DIRECTION $\langle 1,-2,8\rangle$ WHICH IS ORTHOGONAL TO PLANE!

NORMAL: $\langle 1,-2,8\rangle$

$$\boxed{(x-3) - 2(y-2) + 8(z-1) = 0}$$

3. Find the equation of the plane that is parallel to $5x - 3y + 2z = 6$ and goes through the point P(4,-1,2).




1 POINT: $(4,-1,2)$

2 $\langle 5,-3,2\rangle$ IS ORTHOGONAL TO BOTH PLANES!

$$\boxed{5(x-4) - 3(y+1) + 2(z-2) = 0}$$

4. Find the equation of the plane that contains the intersecting lines
$x = 4 + t_1, y = 2t_1, z = 1 - 3t_1$ and
$x = 4 - 3t_2, y = 3t_2, z = 1 + 2t_2$.

FIND 3 PTS:

$t_1 = 0 \Rightarrow (4,0,1)$ $\quad t_1 = 1 \Rightarrow Q(5,2,-2)$

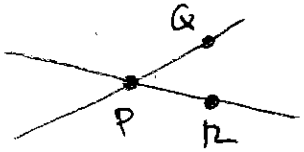
$t_2 = 0 \Rightarrow P(4,0,1)$ $\quad t_2 = 1 \Rightarrow R(1,3,3)$

$\boxed{1}$ POINT: $(4,0,1)$

$\boxed{2}$ NORMAL: $\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -3 \\ -3 & 3 & 2 \end{vmatrix}$

$= (4 - -9)\vec{i} - (2 - 9)\vec{j} + (3 - -6)\vec{k}$

$= \langle 13, 7, 9 \rangle$ CHECK ✓




$$\boxed{13(x-4) + 7y + 9(z-1) = 0}$$

5. Find the equation of the plane that is orthogonal to $3x + 2y - z = 4$ and goes through the points P(1,2,4) and Q(-1,3,2).

$\boxed{1}$ POINT: $(1,2,4)$

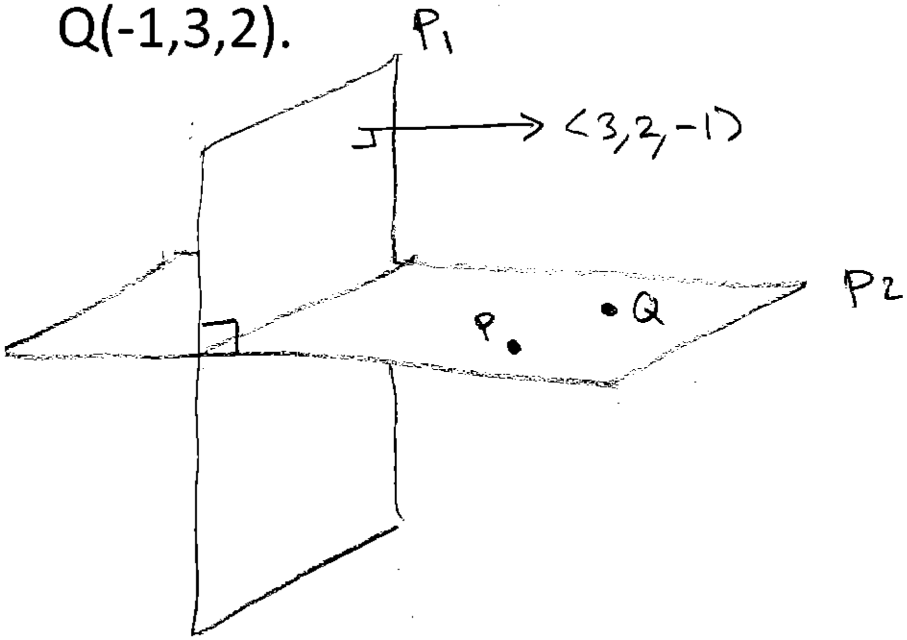
$\boxed{2}$ $\langle 3,2,-1 \rangle$ IS PARALLEL TO THE DESIRED PLANE

$\vec{PQ} = \langle -2, 1, -2 \rangle$ IS ALSO PARALLEL.

$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ -2 & 1 & -2 \end{vmatrix}$

$= (-4 - -1)\vec{i} - (-6 - 2)\vec{j} + (3 - -4)\vec{k}$

$= \langle -3, 8, 7 \rangle$ CHECK ✓




$$\boxed{-3(x-1) + 8(y-2) + 7(z-4) = 0}$$

1. Find the intersection of the line $x = 3t$, $y = 1 + 2t$, $z = 2 - t$ and the plane $2x + 3y - z = 4$.

COMBINE CONDITIONS !!!

$$2(3t) + 3(1+2t) - (2-t) = 4$$
$$6t + 3 + 6t - 2 + t = 4$$
$$13t = 3 \Rightarrow t = 3/13$$
$$x = 9/13, \quad y = 1 + 6/13 = 19/13, \quad z = 2 - 3/13 = \frac{23}{13}$$
$$(x, y, z) = \left(\frac{9}{13}, \frac{19}{13}, \frac{23}{13}\right)$$

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2. Find the intersection of the two lines $x = 1 + 2t_1$, $y = 3t_1$, $z = 5t_1$ and $x = 6 - t_2$, $y = 2 + 4t_2$, $z = 3 + 7t_2$ (or explain why they don't intersect).

COMBINE CONDITIONS!!!

(i) $1 + 2t_1 \overset{?}{=} 6 - t_2 \Rightarrow t_2 = 5 - 2t_1$

(ii) $3t_1 \overset{?}{=} 2 + 4t_2$

$$3t_1 = 2 + 4(5 - 2t_1)$$
$$3t_1 = 2 + 20 - 8t_1$$
$$11t_1 = 22 \quad t_1 = 2$$
$$t_2 = 5 - 2(2) = 1$$

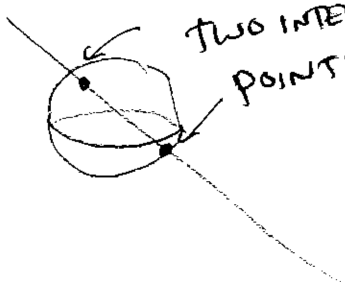
(iii) $5t_1 = 10$, $3 + 7t_2 = 10$ YES!!!

THEY DO INTERSECT WHEN $t_1 = 2$ AND $t_2 = 1$

WHICH GIVES
$$x = 1 + 2(2) = 6 - (1) = 5 \checkmark$$
$$y = 3(2) = 2 + 4(1) = 6 \checkmark$$
$$z = 5(2) = 3 + 7(1) = 10 \checkmark$$

$$\boxed{(5, 6, 10)}$$

3. Find the intersection of the line $x = 2t$, $y = 3t$, $z = -2t$ and the sphere $x^2 + y^2 + z^2 = 16$.



TWO INTERSECTION POINTS

COMBINE CONDITIONS!!!

$$(2t)^2 + (3t)^2 + (-2t)^2 \overset{?}{=} 16$$

$$4t^2 + 9t^2 + 4t^2 = 16$$

$$17t^2 = 16$$

$$t^2 = 16/17 \Rightarrow t = \pm \frac{4}{\sqrt{17}}$$

$$\boxed{\begin{aligned}(x, y, z) &= \left(-\frac{8}{\sqrt{17}}, \frac{-12}{\sqrt{17}}, \frac{8}{\sqrt{17}}\right) \\ \text{AND} \quad &= \left(\frac{8}{\sqrt{17}}, \frac{12}{\sqrt{17}}, -\frac{8}{\sqrt{17}}\right)\end{aligned}}$$

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4. Describe the intersection of the plane $3y + z = 0$ and the sphere $x^2 + y^2 + z^2 = 4$.

PLANE

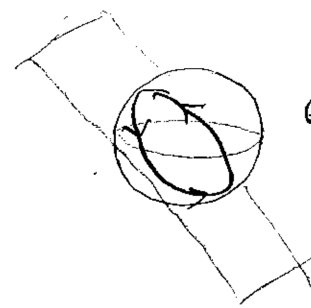
COMBINE CONDITIONS!!!

$$z = -3y$$

$$\Rightarrow \quad x^2 + y^2 + (-3y)^2 = 4$$

$$x^2 + 10y^2 = 4$$

ELLIPSE (WHEN VIEWED FROM ABOVE)



$$x^2 + (\sqrt{10}\,y)^2 = 4$$

ASIDE: PARAMETERIZE

$$\left.\begin{aligned} x &= 2\cos(t) \\ \sqrt{10}\, y &= 2\sin(t) \end{aligned}\right\}$$ DESCRIBES THIS

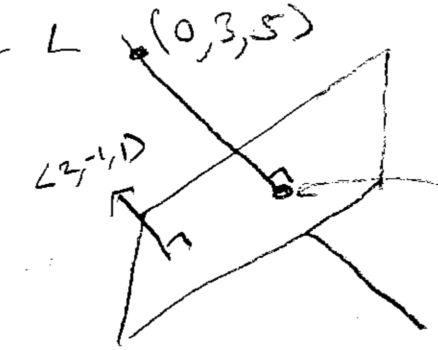
$$\Rightarrow \quad z = -3y = -\frac{6}{\sqrt{10}}\sin(t)$$

$$\boxed{x = 2\cos(t),\ y = \frac{2}{\sqrt{10}}\sin(t),\ z = -\frac{6}{\sqrt{10}}\sin(t)}$$

MOVES AROUND THIS CURVE

Questions directly from old tests:

1. Consider the line thru (0, 3, 5) that is orthogonal to the plane
$$2x - y + z = 20.$$
Find the point of intersection of the line and the plane.




$x = 0 + 2t$
$y = 3 - t$
$z = 5 + t$

INTERSECT

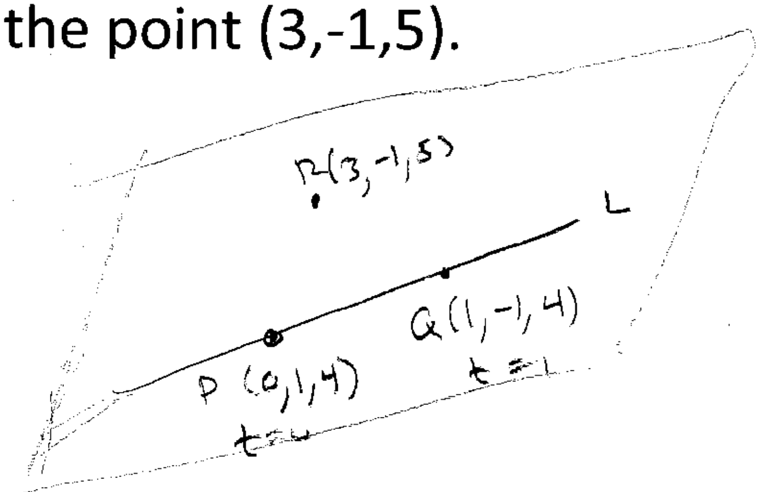
$2(2t) - (3-t) + (5+t) \stackrel{?}{=} 20$

$\Rightarrow 4t - 3 + t + 5 + t = 20 \Rightarrow 6t = 18$
$t = 3$

$x = 6,\ y = 0,\ z = 8$ $\quad \boxed{(6,0,8)}$

ASIDE: DIST FROM (0,3,5) TO (6,0,8) WOULD BE DIST. TO PLANE.

2. Find the equation for the plane that contains the line
$x = t, y = 1 - 2t, z = 4$ and the point (3,-1,5).




3 POINTS!

$\boxed{1}$ POINT (3,-1,5)

$\boxed{2}$ NORMAL

$\vec{n} = \vec{PQ} \times \vec{PR} = \langle 1,-2,0 \rangle \times \langle 3,-2,1 \rangle$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 0 \\ 3 & -2 & 1 \end{vmatrix} = (-2-0)\vec{i} - (1-0)\vec{j} + (-2 - -6)\vec{k}$$
$$= \langle -2,-1,4 \rangle$$

CHECK ✓
$-2+2+0=0$ ✓
$-6+2+4=0$ ✓

$$\boxed{-2(x-3) - (y+1) + 4(z-5) = 0}$$

CHECK